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An Accurate Multi-level Finite Difference Scheme for 1D Diffusion Equations Derived from the Lattice Boltzmann Method

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Abstract An accurate and unconditionally stable explicit finite difference scheme for 1D diffusion equations is derived from the lattice Boltzmann method with rest particles. The system of the lattice Boltzmann equations for the distribution of the number of the fictitious particles is rewritten as a four-level explicit finite difference equation for the concentration of the diffused matter with two parameters. The consistency analysis of the four-level scheme shows that the two parameters which appear in the scheme, the relaxation parameter and the amount of rest particles, can be determined such that the scheme has the truncation error of fourth order. Numerical experiments demonstrate the fourth-order rate of convergence for various combinations of model parameters.

Keywords Lattice Boltzmann method · Multi-level explicit scheme · Accuracy · Stability

1 Introduction

The lattice Boltzmann methods (LBM) have been used for computing numerical solutions of partial differential equations which govern various physical phenomena [1, 2]. The numerical schemes based on the LBM are given as a system of two-level explicit difference equations composed of the distribution functions of fictitious particles for each direction in which the particles move. For advection-diffusion problems Rasin et al. [3] proposed the LB scheme based on a matrix formulation for two-dimensional advection-diffusion equations, Succi et al. [4] formulated a Lax-Wendroff finite difference scheme for the transport of multiple chemical components. R.G.M van der Sman and M.H. Ernst [5] investigated the accuracy of the LB schemes for pure diffusion problems by using the eigenmode analysis.

For one-dimensional advection-diffusion problems Ancona [6] showed that the LB schemes with the velocity model D1Q2 which includes two velocities with speed 1 in opposite directions to each other can be rewritten as the *DuFort–Frankel* scheme [7] which is a second-order three-level difference scheme. This shows that the accuracy of the LB

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schemes based on the model D1Q2 is identical to that of the *DuFort–Frankel* scheme. Thus, if the multi-level difference scheme is derived from the system of difference equations which gives the LB scheme we can investigate the accuracy of the LB scheme through the consistency analysis of the multi-level scheme. Indeed Suga [8] revealed that the accuracy of the LB scheme for two-dimensional advection-diffusion equations with velocity model D2Q4 which is composed of four velocities with speed 1 along coordinate axes is comparable to the *DuFort–Frankel* scheme by deriving the five-level difference scheme for the concentration of the diffused matter. The previous studies show that the LB schemes for pure diffusion problems based on the velocity model without rest particles derive second-order difference schemes.

In this paper, we derive from the LB schemes multi-level explicit difference schemes that are more accurate than second-order schemes for one-dimensional pure diffusion problems. Since the advantage of the LBM is that it provides stable explicit numerical methods, it is expected that an accurate and stable explicit multi-level scheme is obtained. With this aim we introduce a method where the amount of rest particles and the relaxation parameter can be used to obtain fourth order accuracy of the scheme. In the LB schemes for diffusion problems presented in the previous studies [3, 4] the weight coefficients are given as the fixed values. In the LB scheme based on the model D1Q2 for 1D diffusion problems the values of two weight coefficients are determined to be $1/2$ under the condition of momentum conservation [9]. So we study the four-level scheme (more than three-level for the case of the D1Q2 model) derived from the LB schemes with the velocity model D1Q3 [2] which include the velocities in the model D1Q2 and a velocity with speed 0 (rest particles). Two weight parameters, one is for the particles with speed 1 and the other for the rest particles with speed 0, are considered. First we introduce the LB schemes with two parameters: the relaxation parameter ω and the rest particle parameter r (see Sect. 2.1). Next the four-level explicit difference scheme is derived from the LB scheme. The consistency analysis shows that the two parameters can be determined so that the four-level scheme has a truncation error of fourth order. Furthermore we prove that the scheme presented here is unconditionally stable for the parameters which appear in the scheme.

This paper is organized as follows: Sect. 2 describes the LB scheme with the velocity model D1Q3 and derives the four-level explicit finite difference scheme which is equivalent to the LB scheme. In Sect. 3 the accuracy of the four-level scheme is investigated through consistency analysis, and Sect. 4 proves the stability of the scheme. In Sect. 5 bench mark problems are solved and the four-level scheme is compared with the *DuFort–Frankel* scheme. It is shown that the fourth-order convergence rate is achieved for the four-level scheme for various discretizing parameters. Section 6 provides a summary and concludes the paper.

2 The Four-Level Finite Difference Scheme for 1D Diffusion Equations Based on the LB Schemes

We describe the LB schemes with one free parameter based on the velocity model D1Q3 for one-dimensional diffusion equations with constant diffusion coefficient, and derive the four-level explicit finite difference scheme which is equivalent to the LB scheme.

2.1 The LB Scheme

Let $T(x,t)$ denote the concentration of matter which is diffused at time, t, and position, x. Fictitious particles are introduced at each of the mesh points $x = j\Delta x\,(j =$

$\dots, -2, -1, 0, 1, 2, \dots$), and they move with the velocity c_m determined by the D1Q3 model from x to the neighboring mesh point. The D1Q3 model includes the velocities $c_m = m\Delta x/\Delta t\,(m = -1, 0, 1)$. Let $T_m(x,t)$ denote the distribution function of the particles moving with velocity c_m. The time evolution of the distribution function $T_m(x,t)$ is given by the following lattice Boltzmann equation (LBE) based on the Bhatnagar, Gross and Krook (BGK) model [10]:

$$T_m(x + c_m\Delta t, t + \Delta t) = T_m(x,t) - \omega\big(T_m(x,t) - T_m^{\mathrm{eq}}(x,t)\big). \tag{1}$$

On the right hand side of (1) $T_m^{\mathrm{eq}}(x,t)$ is the equilibrium distribution function of the particles which move with velocity c_m. The change in the distribution function produced by the collision of particles is approximated by the second term on the right hand side of (1). The parameter ω is called the relaxation parameter; it is restricted to $0 < \omega < 2$ by stability reasons. Then, the approximate solution of $T(x,t)$, $\bar{T}(x,t)$, is given as

$$\bar{T}(x,t) = \sum_m T_m(x,t) = \sum_m T_m^{\mathrm{eq}}(x,t). \tag{2}$$

In this paper $T_m^{\mathrm{eq}}(x,t)$, $(m = -1, 0, 1)$, are determined as to satisfy (2) and the following conditions [9]:

$$T_m^{\mathrm{eq}}(x,t) = w_m T(x,t), \quad w_m > 0, \tag{3}$$

$$\sum_m c_m T_m^{\mathrm{eq}}(x,t) = 0. \tag{4}$$

Thus, the weights w_m are given as

$$w_{-1} = w_1 = \frac{1-r}{2}, \quad w_0 = r,\ 0 < r < 1, \tag{5}$$

where r is the rest particle free parameter. For $r = 0$ (no rest particles) one recovers the diffusion model based on D1Q2. Applying the Chapman–Enskog expansion [9] yields the following diffusion equation for T from the LBE and the equilibrium distribution functions given by (1) and (3), respectively:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{6}$$

where κ is the diffusion coefficient given as

$$\kappa = (1-r)\frac{(2-\omega)}{2\omega}\frac{\Delta x^2}{\Delta t}. \tag{7}$$

Now we let $T_{m_j}^n$ denote $T_m(j\Delta x, n\Delta t)$. We note that the subscript of T, m_j, combines information about the channel or direction of propagation ($m = -1, 0$, or 1) and location (j denotes a grid node). Substituting (3) into (1), replacing T with $\bar{T}$ in (2) and using (3) gives the following system of finite difference equations for the distribution functions T_m:

$$T_{m_j}^{n+1} = (1-\omega)T_{m_{j-m}}^n + \omega \times w_m \sum_{k=-1,0,1} T_{k_{j-m}}^n \quad (m = -1, 0, 1). \tag{8}$$

Then, the approximate solution $\bar{T}(j\Delta x, n\Delta t)$ denoted by $\bar{T}_j^n$ is calculated as

$$\bar{T}_j^n = \sum_{m=-1,0,1} T_{m_j}^n. \tag{9}$$

The LB scheme based on the velocity model D1Q3 is given by (8) and (9).

2.2 The Multi-level Scheme for $\bar{T}_j^n$

The system of the difference equations given by (8) is the scheme for the distribution function T_m. Now we will derive the finite difference equation for $\bar{T}$. From (8) and (9) we have

$$\begin{aligned}\bar{T}_j^{n+1} &= \sum_{m=-1,0,1}\left(\Omega T_{m_{j-m}}^n + \omega\times w_m \sum_{k=-1,0,1} T_{k_{j-m}}^n\right)\\ &= \sum_{m=-1,0,1}\left((\Omega+\omega\times w_m)\bar{T}_{j-m}^n - \Omega\sum_{k\neq m} T_{k_{j-m}}^n\right),\end{aligned}$$

where $\Omega = 1-\omega$. Substituting (8) and (9) into $T_{k_{j-m}}^n$ in the above equation leads to

$$\bar{T}_j^{n+1} = \sum_{m=-1,0,1}(\Omega+\omega\times w_m)\bar{T}_{j-m}^n - \Omega\sum_{m=-1,0,1}\left(\left(\Omega+\omega\sum_{k\neq m} w_k\right)\bar{T}_{j-m}^{n-1} - \Omega T_{m_{j-m}}^{n-1}\right).$$

Rewriting $T_{m_{j-m}}^{n-1}$ in the same way we have

$$\begin{aligned}\bar{T}_j^{n+1} = \sum_{m=-1,0,1}&(\Omega+\omega\times w_m)\bar{T}_{j-m}^n - \Omega\sum_{m=-1,0,1}\left(\Omega+\omega\sum_{k\neq m} w_k\right)\bar{T}_{j-m}^{n-1}\\ &+\Omega^2\left(\Omega\sum_{m=-1,0,1} T_{m_j}^{n-2} + \omega\bar{T}_j^{n-2}\sum_{m=-1,0,1} w_m\right).\end{aligned}$$

Using (5) and (9) the right hand side of the above equations is rewritten as

$$\begin{aligned}\bar{T}_j^{n+1} = &\sum_{m=-1,0,1}(\Omega+\omega\times w_m)\bar{T}_{j-m}^n\\ &-\Omega\left(\left(\Omega+\frac{1+r}{2}\omega\right)\bar{T}_{j+1}^{n-1} + (\Omega+(1-r)\omega)\bar{T}_j^{n-1} + \left(\Omega+\frac{1+r}{2}\omega\right)\bar{T}_{j-1}^{n-1}\right)\\ &+\Omega^2\left(\Omega\bar{T}_j^{n-2}+\omega\bar{T}_j^{n-2}\right).\end{aligned}$$

Since $\Omega = 1-\omega$ we finally obtain the following four-level explicit scheme for $\bar{T}_j^n$ which is equivalent to the LB scheme (8) and (9):

$$\begin{aligned}\bar{T}_j^{n+1} &= \alpha_1\bar{T}_{j+1}^n + \alpha_2\bar{T}_j^n + \alpha_1\bar{T}_{j-1}^n\\ &\quad+\beta_1\bar{T}_{j+1}^{n-1} + \beta_2\bar{T}_j^{n-1} + \beta_1\bar{T}_{j-1}^{n-1} + \Omega^2\bar{T}_j^{n-2},\end{aligned} \tag{10}$$

$$\alpha_1 = \Omega + a\omega, \qquad \alpha_2 = \Omega + (1-2a)\omega, \tag{11}$$

$$\beta_1 = -(\Omega+(1-a)\omega)\Omega, \qquad \beta_2 = -(\Omega+2a\omega)\Omega, \tag{12}$$

where

$$a = \frac{1-r}{2}, \quad 0 < a < 1/2. \tag{13}$$

3 Accuracy

Here the accuracy of the four-level scheme described in Sect. 2.2 is studied through consistency analysis. Substitution of the exact solution T into (10) and expansion of the terms as the Taylor series about the point $x = j\Delta x$ at time $t = n\Delta t$ gives

$$\begin{aligned}\left[\frac{\partial T}{\partial t}\right]_j^n &= \frac{a(2-\omega)}{\omega}\frac{\Delta x^2}{\Delta t}\left[\frac{\partial^2 T}{\partial x^2}\right]_j^n + \frac{a(2-\omega)}{12\omega}\frac{\Delta x^4}{\Delta t}\left[\frac{\partial^4 T}{\partial x^4}\right]_j^n \\ &\quad - \frac{\beta_1 \Delta x^2 \Delta t}{\omega^2}\left[\frac{\partial^3 T}{\partial x^2 \partial t}\right]_j^n + \left(2\Omega^2 - \frac{1}{2}\right)\frac{\Delta t^2}{\omega^2}\left[\frac{\partial^2 T}{\partial t^2}\right]_j^n + \cdots. \end{aligned}\tag{14}$$

Using (14) and the relations:

$$\left[\frac{\partial^2 T}{\partial t^2}\right]_j^n = \kappa^2\left[\frac{\partial^4 T}{\partial x^4}\right]_j^n, \qquad \left[\frac{\partial^3 T}{\partial x^2 \partial t}\right]_j^n = \kappa\left[\frac{\partial^4 T}{\partial x^4}\right]_j^n,$$

we have

$$\begin{aligned}\left[\frac{\partial T}{\partial t}\right]_j^n &= \kappa\left[\frac{\partial^2 T}{\partial x^2}\right]_j^n - \frac{\kappa \Delta x^2}{2\omega^2}\left(a(\omega^2 - 8\omega + 8) - \frac{\omega^2 + 12\omega - 12}{6}\right)\left[\frac{\partial^4 T}{\partial x^4}\right]_j^n \\ &\quad + O(\Delta x^4). \end{aligned}\tag{15}$$

Now if we set $s = \kappa\Delta t/\Delta x^2$ then from (7) and (13)

$$s = \frac{a(2-\omega)}{\omega}. \tag{16}$$

Thus for given s, if we can find the parameters a and ω satisfying (16) and the equation:

$$a(\omega^2 - 8\omega + 8) - \frac{\omega^2 + 12\omega - 12}{6} = 0, \tag{17}$$

then the scheme with a truncation error of fourth order is obtained. The amount of rest particles r for our scheme is calculated by substituting the value of the parameter a into (13). Figure 1 plots the relaxation parameters ω and the rest particle parameter r satisfying (13), (16) and (17) as the function of the discretizing parameter s ($0 < s \le 1$). This figure shows that for given s the relaxation parameter ω and the amount of rest particles r are found in the intervals $0.3 \le \omega \le 1.1$ and $0.48 \le r \le 0.8$, respectively.

4 Stability

In this section we prove that the four-level scheme presented in this paper is stable for $0 < a < 1/2$ and $0 < \omega < 2$. We begin by considering the LB scheme for the case of $0 < \omega \le 1$.

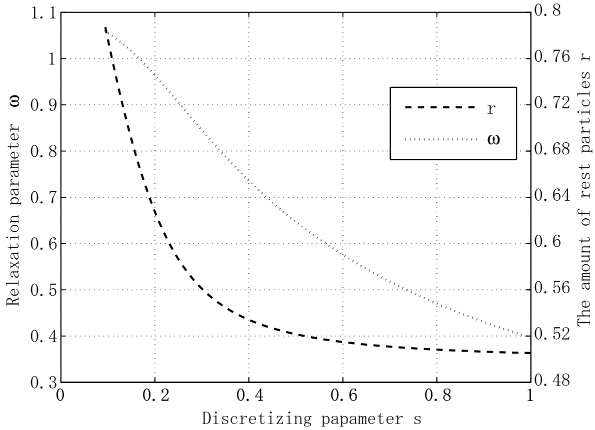

Fig. 1 Plots of the relaxation parameter ω and the amount of rest particles r which satisfy the condition that the four-level scheme has a truncation error of fourth order as the function of the discretizing parameter s

Taking the discrete Fourier transform [11] of $T^n_{m_j}$ in the scheme given by (8) leads to the following matrix equation:

$$\hat{\boldsymbol{U}}^{n+1}_j = \boldsymbol{G}(\boldsymbol{\theta}, \boldsymbol{\omega})\hat{\boldsymbol{U}}^n_j, \tag{18}$$

where vector $\hat{\boldsymbol{U}}^n_j$ consists of the discrete Fourier transform of $T^n_{m_j}$ $(m = -1, 0, 1)$ and $\boldsymbol{G}$ is called the amplification matrix of the scheme given as

$$G(\theta, \omega) = \begin{pmatrix} \Omega + (1-2a)\omega & (1-2a)\omega & (1-2a)\omega \\ a\omega e^{-i\theta} & (\Omega + a\omega)e^{-i\theta} & a\omega e^{-i\theta} \\ a\omega e^{i\theta} & a\omega e^{i\theta} & (\Omega + a\omega)e^{i\theta} \end{pmatrix} \quad (-\pi \le \theta \le \pi). \tag{19}$$

A sufficient condition for the numerical scheme with the amplification matrix $\boldsymbol{G}(\boldsymbol{\theta}, \boldsymbol{\omega})$ to be stable is that the maximum absolute value of the eivenvalue of $\boldsymbol{G}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is less than 1 for θ, $-\pi \le \theta \le \pi$ [12]. Since $0 < a < 1/2$ and $0 < \omega \le 1$ the upper bound of the absolute value of the eigenvalue of the matrix $\boldsymbol{G}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is 1 due to the *Gerschgorin* disc theorem [13]. It shows that the scheme is stable.

Next, we consider the case $1 < \omega < 2$ by applying the discrete Fourier transform to the four-level scheme given by (10). Replacing $\bar{T}^n_j$ by $\lambda^n e^{i\theta j}$ $(-\pi \le \theta \le \pi)$ in (10) we have

$$\xi_3\lambda^3 + \xi_2\lambda^2 + \xi_1\lambda + \xi_0 = 0, \tag{20}$$

where

$$\begin{aligned} \xi_0 &= -\Omega^2, \\ \xi_1 &= \Omega(1 - (1-2a)\omega + 2(1 - a\omega)\cos\theta), \\ \xi_2 &= -(\Omega + (1-2a)\omega + 2(\Omega + a\omega)\cos\theta), \\ \xi_3 &= 1. \end{aligned} \tag{21}$$

The stability condition of the scheme is $|\lambda| \le 1$ for $-\pi \le \theta \le \pi$. So we use the Schur-Cohn algorithm [14] to show that all roots of the above cubic equation lie inside the unit disk for

ω $(1 < \omega < 2)$. We set

$$\begin{aligned}
\eta_0 &= \xi_0^2 - \xi_3^2, \\
\eta_1 &= \xi_0\xi_1 - \xi_3\xi_2, \\
\eta_2 &= \xi_0\xi_2 - \xi_3\xi_1, \\
\zeta_0 &= \eta_0^2 - \eta_2^2, \\
\zeta_1 &= \eta_0\eta_1 - \eta_2\eta_1.
\end{aligned} \tag{22}$$

We can easily see that $\eta_0 < 1$ since $|\Omega| = |1-\omega| < 1$. Thus, if $\zeta_0 > 0$ and $\zeta_0^2 > \zeta_1^2$ then all the roots λ satisfy $|\lambda| < 1$ from the theorem concerning the number of roots λ which satisfy $|\lambda| < 1$ [14]. First, we will show $\zeta_0 > 0$. We can rewrite ζ_0 as

$$\zeta_0 = (\Omega+1)^2(\Omega-1)^2((\Omega+1)^2 - 4(a+(1-a)\cos\theta)^2).$$

Thus we can evaluate ζ_0 as $\zeta_0 \geq (\Omega^2-1)^4 > 0$.

Next, in order to show $\zeta_0^2 > \zeta_1^2$ we rewrite $\zeta_0^2 - \zeta_1^2$ as

$$\zeta_0^2 - \zeta_1^2 = (\eta_0 - \eta_2)^2\left(\xi_0^2 - \xi_3^2\right)\Xi_1\Xi_2,$$

where

$$\Xi_1 = \sum_{k=0}^{3}\xi_k, \qquad \Xi_2 = \sum_{k=0}^{3}(-1)^k\xi_k. \tag{23}$$

Then, we have

$$\Xi_1 = 2a\omega(2-\omega)(1-\cos\theta) \geq 0.$$

In order to evaluate Ξ_2 we obtain

$$\Xi_2 = -(2-\omega)^2 - (1-2a)\omega^2 - 2(a\omega^2 - 2\omega + 2).$$

For $1 < \omega \leq (1-\sqrt{2a})/a$ we have

$$\Xi_2 \leq \omega(1-\omega) < 0,$$

and for $(1-\sqrt{2a})/a \leq \omega < 2$

$$\Xi_2 \leq -2(\omega-2)^2 < 0.$$

Thus $\Xi_2 < 0$ for $1 < \omega < 2$. So we have $\zeta_0^2 - \zeta_1^2 > 0$.

5 Numerical Experiments

We investigate the accuracy of the schemes by solving the benchmark problem described in Fletcher [15]. The initial condition and the boundary condition for the diffusion equation are given as

$$\begin{aligned}
T(x,0) &= 0, \quad 0 \leq x \leq 1, \\
T(0,t) &= T(1,t) = 100, \quad t > 0.
\end{aligned} \tag{24}$$

Table 1 The values of the *RMS* error calculated at $t = n_e \Delta t$ and the convergence rate R

Scheme	s	r	ω	$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.025$	R
D-F	$\sqrt{\frac{1}{12}}$			2.81×10^{-3}	1.73×10^{-4}	1.09×10^{-5}	15.9
D1Q3-4L	$\sqrt{\frac{1}{12}}$	0.562	0.859	5.73×10^{-3}	3.57×10^{-4}	2.25×10^{-5}	15.9
D-F	0.3			1.39×10^{-2}	3.95×10^{-3}	1.04×10^{-3}	3.8
D1Q3-4L	0.3	0.560	0.845	6.84×10^{-3}	4.22×10^{-4}	2.65×10^{-5}	15.9
D-F	0.5			4.07×10^{-1}	9.58×10^{-2}	2.63×10^{-2}	3.8
D1Q3-4L	0.5	0.522	0.647	6.39×10^{-2}	3.55×10^{-3}	2.28×10^{-4}	12.9
D-F	0.75			1.24	3.03×10^{-1}	7.47×10^{-2}	4.0
D1Q3-4L	0.75	0.510	0.493	4.62×10^{-1}	1.97×10^{-2}	1.21×10^{-3}	16.3
D-F	1			2.37	5.75×10^{-1}	1.44×10^{-1}	4.0
D1Q3-4L	1	0.506	0.397	5.89	6.69×10^{-2}	3.92×10^{-3}	17.1

In order to estimate the accuracy of the schemes we calculate the *RMS* error at $t = n\Delta t$:

$$RMS = \sqrt{\frac{\sum_{j=1}^{N}(T(j\Delta x, n\Delta t) - \bar{T}(j\Delta x, n\Delta t))^2}{N}}, \tag{25}$$

where N is the number of the mesh points on $0 < x < 1$ and the exact solution T is calculated in terms of the Fourier series. We set $\kappa = 0.01$ and $\Delta x = 0.1,\ 0.05,\ 0.025$. The bench mark problem is solved by the four-level scheme given by (10) to (12) (hereafter referred to as the D1Q3-4L scheme) and the *DuFort–Frankel* scheme (hereafter referred to as the D-F scheme) for the cases of $s = \sqrt{1/12},\ 0.3,\ 0.5,\ 0.75,\ 1$ and Δt is calculated as $\Delta t = s\Delta x^2/\kappa$ for each s. For $s = \sqrt{1/12}$ the *DuFort–Frankel* scheme has a truncation error of fourth order [15]. The numerical solutions are calculated at time step $n\Delta t$ $(n = 1, 2, \ldots, n_e)$, where n_e satisfies $n_e\Delta t \le T_{\max} < (n_e + 1)\Delta t$, and we set $T_{\max} = 9$. The values of the *RMS* error of the numerical solutions calculated at $t = n_e\Delta t$ by each scheme are summarized in Table 1. The convergence rate of the numerical solutions is estimated by the ratio of the *RMS* error given as

$$R = RMS_{\Delta x=0.05}/RMS_{\Delta x=0.025}, \tag{26}$$

where $RMS_{\Delta x=0.05}$ and $RMS_{\Delta x=0.025}$ show the *RMS* error of the numerical solutions for the cases of $\Delta x = 0.05$ and $\Delta x = 0.05$ respectively. The values of the convergence rate R are also shown in Table 1.

We can see from the table that the D1Q3-4L scheme demonstrates the fourth-order convergence rate for each s, and the D1Q3-4L scheme is more accurate than the D-F scheme except for the case of $s = \sqrt{1/12}$ for which the fourth-order rate of convergence is achieved for the D-F scheme and the case of $s = 1$ and $\Delta x = 0.1$.

6 Summary and Conclusion

A four-level explicit finite difference scheme for 1D diffusion equations with constant diffusion coefficient is derived by rewriting the LB scheme where the D1Q3 velocity model with zero velocity (rest particles) is used and the equilibrium distribution functions of the fictitious particles are expressed in terms of the free parameter. Consistency analysis shows that the free parameter and the relaxation parameter ω can be determined so that the four-level difference scheme has a truncation error of fourth order for given discretizing parameter s. Furthermore, it is proved that the scheme is unconditionally stable. Numerical experiments show that our scheme produces accurate numerical solutions and demonstrate the fourth-order rate of convergence for given discretizing parameter s.

Since our scheme is a four-level scheme, we need four vectors to store the numerical solutions for four time-levels. So it requires more computational resources than the FTCS (a forward difference and a centered difference are used for the time and the spatial derivatives, respectively) scheme or the D-F scheme. At the same time it is shown from the results in this study that the explicit scheme presented here is more accurate than the D-F scheme and achieves the fourth-order rate of convergence for the cases of $s \le 1$. The general three-level fourth-order explicit scheme [15] has a more restrictive stability limitation on s ($s \le 0.34$) than our scheme. Although there are some fourth-order implicit schemes [15] which are unconditionally stable, they require much more computational time especially for using the LU factorization.

Thus it is indicated that our scheme is efficient and produces stable and accurate numerical solutions for various diffusion problems. Furthermore the LBM with the rest particles allows us to give explicit numerical schemes which are as practical as implicit schemes for numerical simulations for diffusion equations.

Work left for the future includes the application of our method to 2D and 3D diffusion equations. In order to derive fourth-order LB schemes in a higher dimension, the LBM with the multi-speed velocity model will be useful, in which different free parameters will be assigned for different values of the speed. The LB schemes for multi-dimensional problems with NV velocities we need $NV + 1$ vectors to store the values at the grid points; NV vectors to store the values of the distribution functions and one vector for the numerical solutions of the diffusion equation. The implicit schemes for multi-dimensional problems require $NP \times NB$ arrays where NP and NB are the number of grid points and the band width of the coefficient matrix of the system of the linear equations derived by discretizing the diffusion equation, respectively. Since $NP \times NB \gg NV + 1$ for most multi-dimensional practical applications, the LB schemes will be much more efficient than the implicit schemes.

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